Name: ___________________________

Directions: **Work only on this sheet** (on both sides, if needed); do not turn in any supplementary sheets of paper. There is actually plenty of room for your answers, as long as you organize yourself BEFORE starting writing.

SHOW YOUR WORK!

1. (20) Just above (10.39), p.318, supposed it said, "The nodes collectively will repeatedly cycle through idle and busy periods, termed I and B periods," instead of defining I and B in terms of just node 0. Here we are in an I period if no nodes are active, and in a B period if at least one node is active. How would (10.39) change?

2. (20) Exercise 20, Chapter 5, p.155.

3. Consider the board game in Sec. 2.8, pp.14-15. This can be modeled as a Markov chain, with state space {0,1,2,...,7}.

(a) (10) Find p_{67}.

(b) (10) Find the long-run fraction of turns in which you get a bonus roll, expressed as a function of π. Note: If you roll and hit 3, and then roll a second time for the bonus, that still only counts one turn, not two.

4. The moment generating function of the random pair (X,Y) is defined by

$$m_{X,Y}(u,v) = E[e^{uX+vY}]$$

(a) (10) For the density (5.17), p.105, find $m_{X,Y}(u,v)$. Express your answer in integral form.

(b) (10) For a general random pair (X,Y), express Cov(X,Y) in terms of moment generating functions.

5. (20) Consider a Markov chain $\{X_n\}$. Let X_0 have the distribution π (an example is discussed in Sec. 10.1.2.4). Show that "given the present, the past and the future are independent," in the sense that for $i > 0$, X_{i-1} and X_{i+1} are independent, given X_i.

Solutions:

1. Again I will have a geometric distribution. However, the "success probability" (the parameter p in (3.66)) changes. "Success" here will be that at least one of the n nodes becomes active, i.e. generates a message to send. This occurs with probability

$$1 - P(\text{no one generates a message}) = 1 - (1-q)^n$$

So,

$$E(I) = \frac{1}{[1-(1-q)^n]}$$

2. From the Law of Total Variance,

$$Var(N) = E[Var(N|L)] + Var[X(N|L)]$$

Since the conditional distribution of N given L is Poisson with parameter L, $Var(N|L) = E(N|L) = L$. The result follows.

3.a We can go from square 6 to square 7 either directly, by rolling a 1, or via a bonus, by rolling a 5 to get to 3, then rolling a 4 to get to 7. The probability of this is

$$\frac{1}{6} + (\frac{1}{6})^2 = \frac{7}{36}$$

3.b We can only get a bonus from squares 0 (by rolling a 3), 1 (by rolling a 2), 2, 5, 6 and 7. It's impossible from square 4, and we are never on square 3 anyway. So,

$$P(\text{get a bonus in turn i}) = \sum_s P(X_{i-1} = s) \cdot P(\text{get a bonus turn}|X_{i-1} = s) = \frac{1}{6} \cdot (\pi_0 + \pi_1 + \pi_2 + \pi_5 + \pi_6 + \pi_7)$$

4.a Use (5.15):

$$m_{X,Y}(u,v) = E[e^{uX+vY}] = \int_0^1 \int_0^t e^{us+vt} 8st\ ds\ dt$$

4.b For example, to get E(XY), note that

$$\frac{\partial^2}{\partial u \partial v} E[e^{uX+vY}] = E[XY e^{uX+vY}]$$

Thus

$$E(XY) = \frac{\partial^2}{\partial u \partial v} m_{X,Y}(u,v)|_{u=v=0}$$

5. We must show that

$$P(X_{i-1} = k \text{ and } X_{i+1} = m | X_i = n) = P(X_{i-1} = k | X_i = n) \cdot P(X_{i+1} = m | X_i = n) \tag{1}$$

But the left-hand side of (1) is

$$\frac{P(X_{i-1} = k \text{ and } X_{i+1} = m \text{ and } X_i = n)}{P(X_i = n)} = \frac{\pi_k p_{kn} p_{nm}}{\pi_n} \tag{2}$$

while the two factors in the right-hand side are

$$\frac{P(X_{i-1} = k \text{ and } X_i = n)}{P(X_i = n)} = \frac{\pi_k p_{kn}}{\pi_n} \tag{3}$$

and

$$p_{nm} \tag{4}$$

Multiplying (3) by (4), we find that the product matches (2)! So we're done.