Name: ___________________________

Directions: **Work only on this sheet** (on both sides, if needed); do not turn in any supplementary sheets of paper. There is actually plenty of room for your answers, as long as you organize yourself BEFORE starting writing.

SHOW YOUR WORK!

1. Suppose that the conditional distribution of Y given X (the latter a scalar, i.e. we have just one predictor) is Poisson with parameter $\beta_1 X$. We wish to estimate β_1 from our data, $(X_1, Y_1), ..., (X_n, Y_n)$, with the n pairs being assumed to be i.i.d.

(a) (25) Find the conditional Maximum Likelihood estimator of β_1. (There is a closed-form solution.)

(b) (25) Give formulas for unconditional and conditional approximate 95% confidence intervals for β_1. **Justify your answers!**

2. (50) Consider the linear regression model in Chapter 9, but with the following change. Instead of the homogeneous-variance model (9.32), suppose that

$$Var(Y_i|X_i) = \sigma_i^2 \tag{1}$$

where the σ_2^2 are <u>known</u> positive quantities.

In that case, (9.25) is still a reasonable estimator—it will still be consistent, for instance—but (9.38) would be invalid and thus not usable for confidence intervals and significance tests. Moreover, (9.25) has certain optimality properties,[1] so it would be good to find a way to still use (9.25) in this heterogeneous-variance setting.

Toward that end, you will transform the original problem into a new one that satisfies (9.25). Do this by finding a constant matrix A such that the new random vector W = AV satisfies the condition (9.32). Then use (9.25) on W, eventually transforming back so that we have an optimal estimator $\hat{\beta}$ of β of the original problem, complete with covariance matrix for that $\hat{\beta}$. Remember, all your final quantities will have the σ_i in them, but that's all right because they are assumed known.

Note: Matrix quantities must be simplified to the extent possible.

Solution:

1. The (conditional) likelihood function is

$$L = \Pi_{i=1}^n \left[\frac{e^{-\beta_1 X_i} (\beta_1 X_i)^{Y_i!}}{Y_i!} \right] \tag{2}$$

So set

$$l = \ln(L) = \sum_{i=1}^n [-\beta_1 X_i + Y_i(\ln(\beta_1) + \ln(X_i) - \ln(Y_i))] \tag{3}$$

Then setting $0 = \frac{dl}{d\beta_1}$ and solving, we get

$$\hat{\beta}_1 = \frac{\hat{Y}}{\hat{X}} \tag{4}$$

For constructing the unconditional confidence interval for β_1, we know from the application of Slutsky's Theorem in Section 7.4.3 that we can treat as a constant. Our confidence interval is then

$$\frac{\hat{Y}}{\hat{X}} \pm 1.96 \frac{s_Y}{\bar{X}} \tag{5}$$

s_Y^2 is the sample variance of the Y_i.

For the conditional confidence, note that by the Poisson property,

[1] Minimum variance among all unbiased estimators.

$$Var(Y_i|X_i) = E(Y_i|X_i) = \beta_1 X_i \tag{6}$$

Then

$$\begin{aligned} Var(\bar{Y}| \text{ the } X_i) &= Var\left(\frac{Y_1 + ... + Y_n}{n} \middle| X_i \right) &(7)\\ &= \frac{1}{n^2} \sum_{i=1}^{n} Var(Y_i|X_i) &(8)\\ &= \frac{1}{n^2} \sum_{i=1}^{n} \beta_1 X_i &(9)\\ &= \beta_1 \bar{X}/n &(10) \end{aligned}$$

So from (4) we have

$$Var(\hat{\beta}_1| \text{ the } X_i) = \frac{1}{\bar{X}^2} Var(\bar{Y}| \text{ the } X_i) = \beta_1/(n\bar{X}) \tag{11}$$

The conditional standard error of $\hat{\beta}_1$ is then

$$\sqrt{\hat{\beta}_1/(n\bar{X})} = \sqrt{\bar{Y}/(n\bar{X}^2)} \tag{12}$$

2. Set A to be the diagonal matrix whose ith element is $1/\sigma_i$. Then

$$Cov(W) = I \tag{13}$$

That means that (9.32) is satisified, with $\sigma^2 = 1$. (Note that the σ_i are KNOWN. It was wrong to answer that A would have as its ith element $1/\sigma_i$, as that would introduce a new quantity, σ^2 that is not part of the problem.)

Let γ denote the vector of population coefficients for the regression function of W on Q. (9.32) gives us

$$\hat{\gamma} = (q'q)^{-1}q'w \tag{14}$$

Now to get back to the original setting with V instead of W, note that since W = AV, we have

$$\gamma = E(W|Q) = AE(V|Q) = A\beta \tag{15}$$

Thus we set

$$\hat{\beta} = A^{-1}\hat{\gamma} \tag{16}$$

so

$$\hat{\beta} = A^{-1}(Q'Q)^{-1}Q'W = A^{-1}(Q'Q)^{-1}Q'AV \tag{17}$$

Writing that last quantity as RV, we have

$$Cov(\hat{\beta}) = RCov(V)R' = R(A^{-1})^2R' \tag{18}$$

Lots of stuff cancels in that last expression, and eventually we get

$$Cov(\hat{\beta}) = A^{-1}(Q'Q)^{-1}A^{-1} \tag{19}$$