Name: ____________________________

Directions: **Work only on this sheet** (on both sides, if needed); do not turn in any supplementary sheets of paper. There is actually plenty of room for your answers, as long as you organize yourself BEFORE starting writing.

Note: Some problems may require answers to be in the form of numerical expression. An example is:

$$2/7 \cdot 1.39 + \sqrt{29.002} + \int_0^1 0.82t \ dt + (1,2) \begin{pmatrix} 1 & 4 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \end{pmatrix} \tag{1}$$

No variables are allowed in numerical expressions. Also, infinite series do not count either.

1. There is a town with two social groups. Everyone is in exactly one group People arrive from outside town, with exponentially distributed interarrival times at rate α, and join one of the groups with probability 0.5 each. Each person will occasionally switch groups, with one possible "switch" being to leave town entirely. A person's time before switching is exponentially distributed with rate σ; the switch will either be to the other group or to the outside world, with probabilities q and 1-q, respectively. Let the state of the system be (i,j), where i and j are the number of current members in groups 1 and 2, respectively.

Answer in terms of α, λ, τ and π:

(a) Give the balance equation for the state (8,8).

(b) We have just entered state (5,0). What is the mean time until Group 2 becomes nonempty?

(c) The president of Group 1 tells reporter, "We've found over the years that ______________________% of our members come from transfers."

2. Consider a renewal process in which lifetimes have the values 1, 2, 3 or 4, with probability 1/4 each.

(a) Find P[N(3) = 2].

(b) For large integer t, find the probability that the current renewal period began at t-2.

Solutions:

1.

(a)

$$\pi_{(8,8)}(\alpha + 2 \cdot 8 \cdot \sigma) = (\pi_{(9,8)} + \pi_{(8,9)}) \cdot \sigma(1-q) + (\pi_{(9,7)} + \pi_{(7,9)}) \cdot \sigma q + (\pi_{(8,7)} + \pi_{(7,8)}) \cdot 0.5\alpha \tag{2}$$

(b) . The rate of transitions into that group from outside is 0.5α. When the system is in state (i,j), the rate of transitions into group 1 from group 2 is $j\sigma q$, so the overall rate is $\sum_{i,j} \pi_{(i,j)} j\sigma q$. Thus the fraction of new members coming in to group 1 from transfers is

$$\frac{\sum_{i,j} \pi_{(i,j)} j\sigma q}{\alpha + \sum_{i,j} \pi_{(i,j)} j\sigma q} \tag{3}$$

By the way, note that $\sum_{i,j} \pi_{(i,j)} j\sigma q = \sigma q EN$, where N is the number of members of group 1.

2.

(a)

$$P[N(3) = 2] = P(R_3 \leq 3, R_4 > 4) \text{ (note latter condition!)} \tag{4}$$

Then

$$\begin{aligned} P(R_3 \leq 3, R_4 > 4) &= P(R_1 = 1, R_2 = 2, R_3 > 3 \text{ or } R_1 = 1, R_2 = 3 \text{ or } R_1 = 2, R_2 = 3) && (5) \\ &= P(L_1 = 1, L_2 = 1, L_3 > 1 \text{ or } L_1 = 1, L_2 = 2 \text{ or } L_1 = 2, L_2 = 1) && (6) \\ &= \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} && (7) \\ &= \frac{11}{64} && (8) \end{aligned}$$

(b) This is from p.244. We have

$$\pi_2 = \frac{1 - F_L(3)}{EL} = \frac{1 - 0.75}{2.5} = \frac{1}{10} \tag{9}$$