Let's write X as a sum of those 0-1 Bernoulli variables we used in the discussion of the geometric distribution above:

$$X = \sum_{i=1}^{n} B_i \tag{3.88}$$

where B_i is 1 or 0, depending on whether there is success on the i^{th} trial or not. Note that the B_i are indicator random variables, so $EB_i = p$ and $Var(B_i) = p(1-p)$ (Section 3.6).

Then the reader should use our earlier properties of E() and Var() in Sections 3.4 and 3.5 to fill in the details in the following derivations of the expected value and variance of a binomial random variable:

$$EX = E(B_1 + ..., +B_n) = EB_1 + ... + EB_n = np \tag{3.89}$$

and from (3.64),

$$Var(X) = Var(B_1 + ..., +B_n) = Var(B_1) + ... + Var(B_n) = np(1-p) \tag{3.90}$$

Again, (3.89) should make good intuitive sense to you.

3.12.4 Example: Flipping Coins with Bonuses

A game involves flipping a coin k times. Each time you get a head, you get a bonus flip, not counted among the k. (But if you get a head from a bonus flip, that does not give you its own bonus flip.) Let X denote the number of heads you get among all flips, bonus or not. Let's find the distribution of X.

Toward this end, let Y denote the number of heads you obtain through nonbonus flips. Y then has a binomial distribution with parameters k and 0.5. To find the distribution of X, we'll condition on Y.

We will as usual ask, "How can it happen?", but we need to take extra care in forming our sums, recognizing constraints on Y:

- $Y \geq X/2$
- $Y \leq X$
- $Y \leq k$

Keeping those points in mind, we have

$$
\begin{aligned}
p_X(m) &= P(X = m) && (3.91)\\
&= \sum_{i=\text{ceil}(m/2)}^{\min(m,k)} P(X = m \text{ and } Y = i) && (3.92)\\
&= \sum_{i=\text{ceil}(m/2)}^{\min(m,k)} P(X = m|Y = i)\ P(Y = i) && (3.93)\\
&= \sum_{i=\text{ceil}(m/2)}^{\min(m,k)} \binom{i}{m} 0.5^m \binom{k}{i} 0.5^i && (3.94)\\
&= \sum_{i=\text{ceil}(m/2)}^{\min(m,k)} \frac{k!}{m!(i-m)!(k-i)!} 0.5^{m+i} && (3.95)
\end{aligned}
$$

There doesn't seem to be much further simplification possible here.

3.12.5 Example: Analysis of Social Networks

One of the earliest—and now the simplest—modesl of social networks is due to Erdös and Renyi. Say we have n people (or n Web sites, etc.), with $\binom{n}{2}$ potential links between pairs. (We are assuming an undirected graph here.) In this model, each potential link is an actual link with probability p, and a nonlink with probability 1-p, with all the potential links being independent.

One entity of interest is the **degree distribution**, defined as follows. For each node, the number of links connected to it is called the **degree** of the node. Since degree is a random variable, we can ask about its distribution.

Clearly the degree distribution for a single node is binomial with parameters $\binom{n-1}{2}$ and p. But consider k nodes, and the total T of their degrees. Let's find the distribution of T.

That distribution is again binomial, but the number of trials is not $k\binom{n-1}{2}$, due to overlap. There are $\binom{k}{2}$ potential links among these k nodes, and each has $\binom{n-k}{2}$ potential links to the "outside world," i.e. to the remaining n-k nodes. So, the distribution of T is binomial with

$$
k\binom{n-k}{2} + \binom{k}{2} \tag{3.96}
$$

trials and success probability p.