New Computational Approaches to Large/Complex Mixed Effects Models

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A Different Kind of Talk

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- Suggestions on *what* (new) to model, not *how*.

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- Consider the former n_i to be random, i.e. N_i, with their own effects worth studying. (In 2-factor models, N_i are row counts, and have column counts M_i. Etc.)

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    - Enables statistical analysis of N!

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- Research on negative correlation of family size to household income (Berger, 2011).

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- Once we go to two-factor models, add regressors etc., things get even messier, FAST.

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$$Var(W) = E[Var(W|U)] + Var[E(W|U)] \tag{4}$$

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Write $Y = \mu + \alpha + \epsilon$ and, motivated by defining

$$S_i = \sum_{j=1}^{N_i} Y_{ij} \approx N\mu + N\alpha \tag{5}$$

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Finally, replace pop. quantities by sample analogs, and solve.

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Let's add a covariate:

$$Y_{ij} = \beta_0 + X_i \beta_1 + \alpha_i + \epsilon_{ij}, \; i = 1, ..., r, j = 1, ..., N_i \tag{7}$$

$$Y = \beta_0 + X \beta_1 + \alpha + \epsilon \tag{8}$$

$$S = N(\beta_0 + X \beta_1 + \alpha) + \epsilon_1 + ... + \epsilon_N \tag{9}$$

and the matrix equation

$$\mathbb{Y} = \mathbb{B}_0 + \mathbb{X} \beta_1 + \mathbb{A} + \mathbb{G} \tag{10}$$

This is already getting messy even in this form, but much better than the standard way, with all the messy sums.

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- Hence my charactizeration of the method as an algebraic computational device.

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- Due to i.i.d. nature, lends to easy parallelization through "software alchemy" (Matloff, 2016 JSS and references therein).

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- MM is much faster and uses far less memory than MLE.

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- Streamlines algebraic derivations.
- Allows investigation of the effects of the N_i themselves.
- Enables parallelization.